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Wave Momentum Flux Parameter for Coastal Structure Design

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PURPOSE: The Coastal and Hydraulics Engineering Technical Note (CHETN) described herein provides information about a new wave parameter for characterizing wave processes at coastal structures. A description of the parameter is given along with sample calculations for periodic waves and solitary waves. The first application of this physically relevant parameter has been development of new empirical relationships for irregular wave runup on smooth, impermeable slopes described in CHETN-III-68 (Hughes 2003).

COMMON WAVE PARAMETERS: Coastal engineers have established useful design guidance by augmenting theoretical reasoning with empirical coefficients determined from small-scale laboratory testing. Waves are usually included in empirical design relationships via one or more wave parameters considered to be representative of the incident wave condition. Common regular and irregular wave parameters are listed in Table 1.

Table 1 Common Wave and Fluid Parameters	
Regular Wave Parameters	
H = wave height	H_o = deepwater wave height
L = local wavelength	L_o = deepwater wavelength
T = wave period	k = wave number $[= 2\pi/L]$
Irregular Wave Parameters	
H_{mo} = zeroth-moment wave height	H_s = significant wave height $[= H_{1/3}]$
H_{rms} = root-mean-squared wave height	$H_{10\%}$ = 10% of waves are higher
T_p = spectral peak wave period	T_m = mean wave period
L_p = wavelength associated with T_p	L_{op} = deepwater wavelength associated with T_p
L_m = wavelength associated with T_m	L_{om} = deepwater wavelength associated with T_m
Fluid and Other Parameters	
g = gravitational acceleration	ρ = fluid density
μ = coefficient of dynamic viscosity	ν = coefficient of kinematic viscosity
h = water depth	α = beach or structure slope

Sometimes, these wave parameters are combined to form dimensionless variables that may include relevant fluid and other parameters such as those given in Table 1. The more common dimensionless wave parameters that have appeared in coastal structure design guidance are shown in Table 2. Note that relative wave height H/h is the only parameter that does not contain wave period or wavelength. Hence, H/h is the only parameter listed that is applicable to solitary waves.

The dimensionless wave parameters listed in Table 2 strictly pertain to uniform, periodic waves of permanent form (i.e., regular waves); and it is customary to use first-order wave theory to calculate

values for wavelength. These same dimensionless parameters are also used to characterize irregular wave trains by substituting wave heights, wave periods, and wavelengths representative of irregular waves, such as those shown on Table 1.

This CHETN describes a new wave parameter for depicting processes that occur when waves impinge on coastal structures. The new wave parameter has the following attributes:

- The parameter is physically meaningful so it can be incorporated into theoretical models of specific physical processes in a rational way.
- The parameter applies to both periodic waves and nonperiodic waves.
- The parameter spans the range of relative depths from deep water to shallow water.
- The parameter is stable and robust.
- The parameter is easy to estimate so design guidance using the parameter can be programmed into computer spreadsheets or simple programs.

Table 2 Common Dimensionless Wave Parameters	
Parameter	Value
Relative depth	$\frac{h}{L} ; \frac{h}{gT^2} ; kh$
Relative wave height	$\frac{H}{h}$
Wave steepness	$\frac{H}{L} ; \frac{H}{L_o} ; \frac{H}{gT^2}$
Deepwater wave steepness	$\frac{H_o}{L_o} ; \frac{H_o}{gT^2}$
Local Iribarren number	$\xi = \frac{\tan \alpha}{\sqrt{H/L}}$
Deepwater Iribarren number	$\xi = \frac{\tan \alpha}{\sqrt{H_o/L_o}}$

WAVE MOMENTUM FLUX PARAMETER: Surface waves possess momentum that is directed parallel to the direction of wave propagation. When a wave encounters a solid object, momentum is reversed, and the resulting force on the object is equal to the rate of change of momentum, also known as the wave momentum flux. Thus, wave momentum flux is the property of progressive waves most closely related to force loads on coastal structures or any other solid object placed in the wave field. And for this reason wave momentum flux is a compelling wave property for characterizing coastal structure response to wave loading.

The instantaneous flux of horizontal momentum (m_f) across a unit area of a vertical plane oriented parallel to the wave crests is given by

$$m_f(x, z, t) = p_d + \rho u^2 \quad (1)$$

where

x = horizontal direction perpendicular to wave crests

z = vertical direction, positive upward with $z = 0$ at still-water level (swl)

t = time

p_d = instantaneous wave dynamic pressure at a specified position

ρ = water density

u = instantaneous horizontal water velocity at the same specified position

Integrating the instantaneous wave momentum flux over the water depth, i.e.,

$$M_F(x, t) = \int_{-h}^{\eta(x)} (p_d + \rho u^2) dz \quad (2)$$

gives the total wave momentum flux per unit crest length acting through a vertical plane parallel to the crest. If the depth-integrated M_F is also integrated over the wavelength, the result becomes the wave “radiation stress” defined by Longuet-Higgins and Steward (1964). However, there is significant variation of depth-integrated wave momentum flux over a wavelength from large positive values to large negative values. So instead of adopting a mean value which is quite small compared to the range of variation, it is logical when considering the wave force loading on structures to focus on the maximum, depth-averaged wave momentum flux that occurs during passage of a wave, i.e., the maximum of Equation 2, which occurs at the wave crest.

Using Equation 2, maximum depth-integrated wave momentum flux $(M_F)_{\max}$ can be determined for any surface wave form provided the velocity and pressure field can be specified. Therefore, $(M_F)_{\max}$ has the potential of being a unifying wave parameter applicable to both periodic and transient wave types.

ESTIMATING MAXIMUM DEPTH-INTEGRATED WAVE MOMENTUM FLUX: Hughes (in preparation) derived formulas for estimating the maximum depth-integrated wave momentum flux for periodic (regular) waves and solitary waves.

Periodic Waves

Expressions derived for maximum depth-integrated wave momentum flux from linear wave theory do not include that part of the wave above the swl where a significant portion of the wave momentum flux is found. Estimates of $(M_F)_{\max}$ improve using extended-linear theory in which expressions for linear wave kinematics are assumed to be valid in the crest region. However, the wave form is still sinusoidal rather than having peaked crests and shallow troughs typical of nonlinear shoaled waves; and consequently, the theory underpredicts momentum flux under the crest. Hughes (in preparation) calculated values of $(M_F)_{\max}$ for a wide range of nonlinear uniform waves using a numerical technique (Fourier approximation) that optimized the solution to provide the best fit of the fully nonlinear free surface boundary conditions. Results were expressed in terms of the dimensionless parameter

$$\left(\frac{M_F}{\rho g h^2} \right)_{\max}$$

This parameter represents the nondimensional maximum depth-integrated wave momentum flux, and it is referred to as the “wave momentum flux parameter.”

Figure 1 presents the dimensionless wave momentum flux parameter versus relative depth h/gT^2 . The solid lines are lines of constant relative wave height H/h . For a constant water depth, wave period increases toward the left and decreases to the right. The range of relative depths covers most coastal applications.

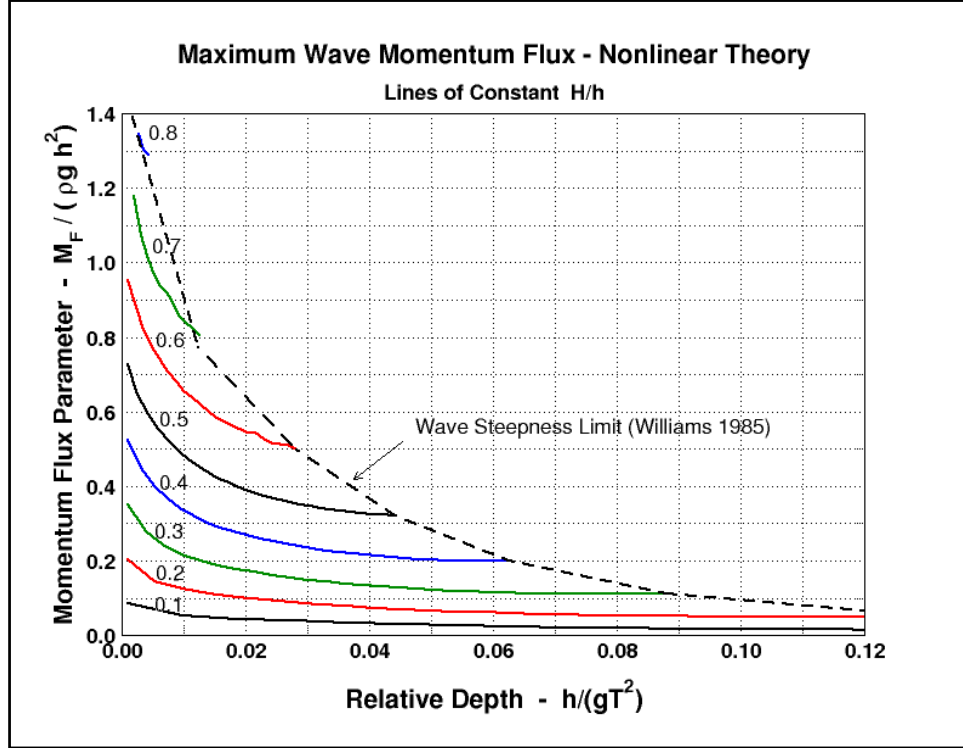


Figure 1. Dimensionless wave momentum flux versus h/gT^2

The dashed line in Figure 1 gives the steepness-limited wave breaking criterion tabulated by Williams (1985) and expressed by Sobey (1998) as the rational approximation

$$\frac{\omega^2 H_{limit}}{g} = c_o \tanh \left(\frac{a_1 r + a_2 r^2 + a_3 r^3}{1 + b_1 r + b_2 r^2} \right) \quad (3)$$

where

$$\begin{aligned} c_o &= 1.0575 \\ a_1 &= 0.7879 \\ r &= \omega^2 h/g \\ a_2 &= 2.0064 \\ a_3 &= -0.0962 \\ b_1 &= 3.2924 \\ b_2 &= -0.2645 \end{aligned}$$

Sobey noted the previous expression has a maximum error of 0.0014 over the range of Williams' table. Williams' (1985) tabulation of limit waves is more accurate than the traditional limit wave steepness given by

$$\frac{H_{limit}}{L} = 0.142 \tanh(kh) \quad (4)$$

which overestimates limiting steepness for long waves and underestimates limiting steepness for short waves.

An empirical equation for estimating the wave momentum flux parameter for finite amplitude steady waves was established using the calculated curves of constant H/h shown in Figure 1. A nonlinear best-fit of a two parameter power curve was performed for each calculated H/h -curve, and the resulting coefficients and exponents for each power curve were also approximated as power curves. The resulting, purely empirical, equation representing the curves of constant H/h shown on Figure 1 is given as

$$\left(\frac{M_F}{\rho g h^2} \right)_{\max} = A_0 \left(\frac{h}{g T^2} \right)^{-A_1} \quad (5)$$

where

$$A_0 = 0.639 \left(\frac{H}{h} \right)^{2.026} \quad (6)$$

$$A_1 = 0.180 \left(\frac{H}{h} \right)^{-0.391} \quad (7)$$

Goodness-of-fit of Equation 5 compared to the computed values given in Figure 1 is shown in Figure 2. For smaller values of nondimensional $(M_F)_{\max}$, there is reasonable correspondence except for the left-most points of each curve (shown below the line of equivalence). This divergence was caused by the power curve tending toward positive infinity as $h/gT^2 \rightarrow 0$. Greater deviation begins to occur for dimensionless $(M_F)_{\max} > 0.6$. Nevertheless, the only poorly fitted curve is for $H/h = 0.8$ which is at or slightly above the limiting steepness for waves on a horizontal seabed. This poor correspondence was likely the result of forcing the numerical computation beyond appropriate limits.

The empirical equation represented by Equation 5, along with Equations 6 and 7, provides an easy method for estimating the nondimensional wave momentum flux parameter for finite amplitude, regular waves. However, most coastal structure design guidance developed in the past 20–25 years has used wave parameters representative of unidirectional irregular wave trains. It is recommended that the wave momentum flux parameter for irregular wave trains be estimated by substituting irregular wave parameters H_{mo} (zeroth-moment wave height) and T_p (peak spectral period) directly into the empirical Equations 5, 6, and 7. Application is simple, and estimates of maximum wave momentum flux should be reasonably representative of the irregular wave train. However, other irregular wave parameters might be used depending on the application. Therefore, before applying any design guidance using the wave momentum flux parameter, it is important to ascertain which wave parameters were used to establish that particular empirical relationship.

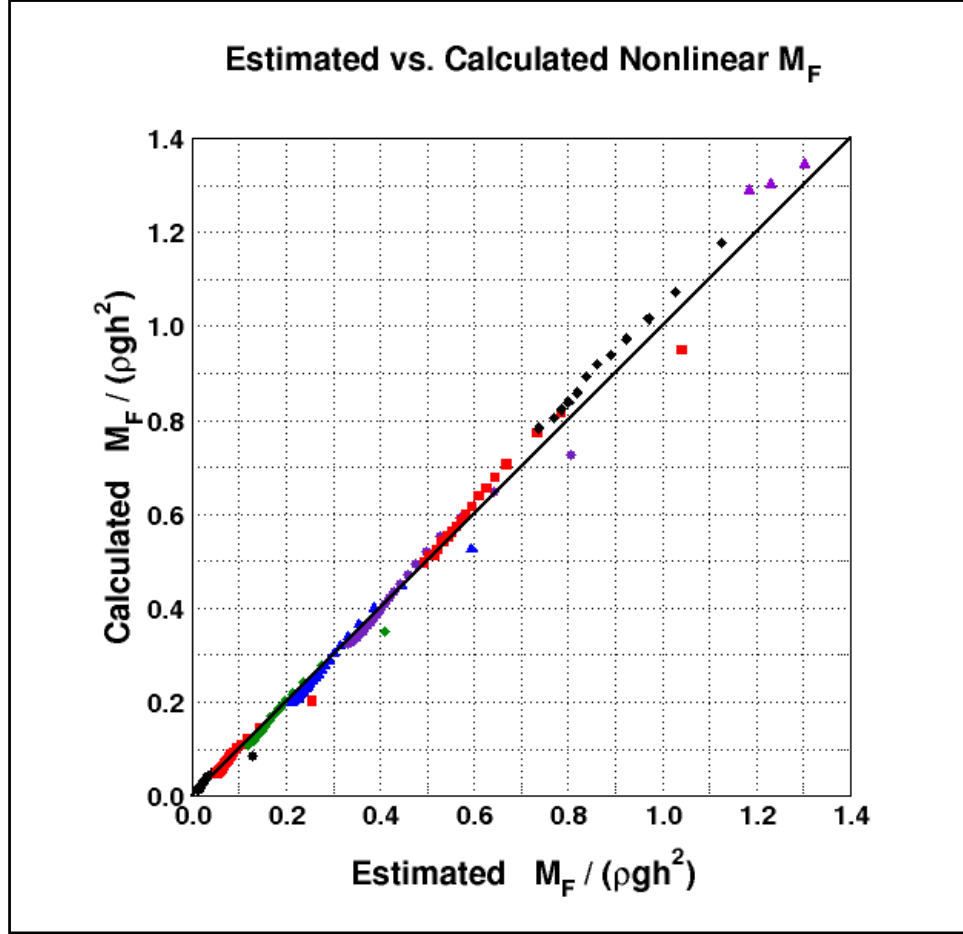


Figure 2. Goodness-of-fit of nonlinear momentum flux empirical equation (Fourier wave theory)

Solitary Waves

An analytical derivation for the wave momentum flux parameter for solitary waves was given by Hughes (in preparation) as

$$\left(\frac{M_F}{\rho g h^2} \right)_{\max} = \frac{1}{2} \left[\left(\frac{H}{h} \right)^2 + 2 \left(\frac{H}{h} \right) \right] + \frac{N^2}{2M} \left(\frac{H}{h} + 1 \right) \left\{ \tan \left[\frac{M}{2} \left(\frac{H}{h} + 1 \right) \right] + \frac{1}{3} \tan^3 \left[\frac{M}{2} \left(\frac{H}{h} + 1 \right) \right] \right\} \quad (8)$$

The parameter is a function of only relative wave height H/h and two coefficients M and N that are typically presented in graphical form. To accommodate calculations, a nonlinear curve fit was applied to the plotted M and N curves to produce the following simple equations that give reasonable values for M and N .

$$M = 0.98 \left\{ \tanh \left[2.24 \left(\frac{H}{h} \right) \right] \right\}^{0.44} \quad (9)$$

$$N = 0.69 \tanh \left[2.38 \left(\frac{H}{h} \right) \right] \quad (10)$$

The empirically-fit equations (solid lines) are plotted along with the data points taken from the *Shore protection manual* (1984) in Figure 3. The variation of maximum nondimensional wave momentum flux for solitary waves as a function of H/h is shown in Figure 4. These values represent the upper limit of nonlinear (Fourier) waves when $h/(gT^2)$ approaches zero.

APPLICATIONS: It is anticipated that the wave momentum flux parameter will prove useful for developing improved semiempirical formulas to describe such wave/structure processes as runup, overtopping, reflection, transmission, and armor stability. As an example, a simple expression for wave runup on plane, impermeable slopes was derived by assuming the weight of water contained in the volume above the still-water line at maximum runup is proportional to the maximum depth-integrated wave momentum flux in the wave as it passes the structure toe. Hughes (in preparation) confirmed this simple hypothesis by establishing empirical runup relationships based on the wave

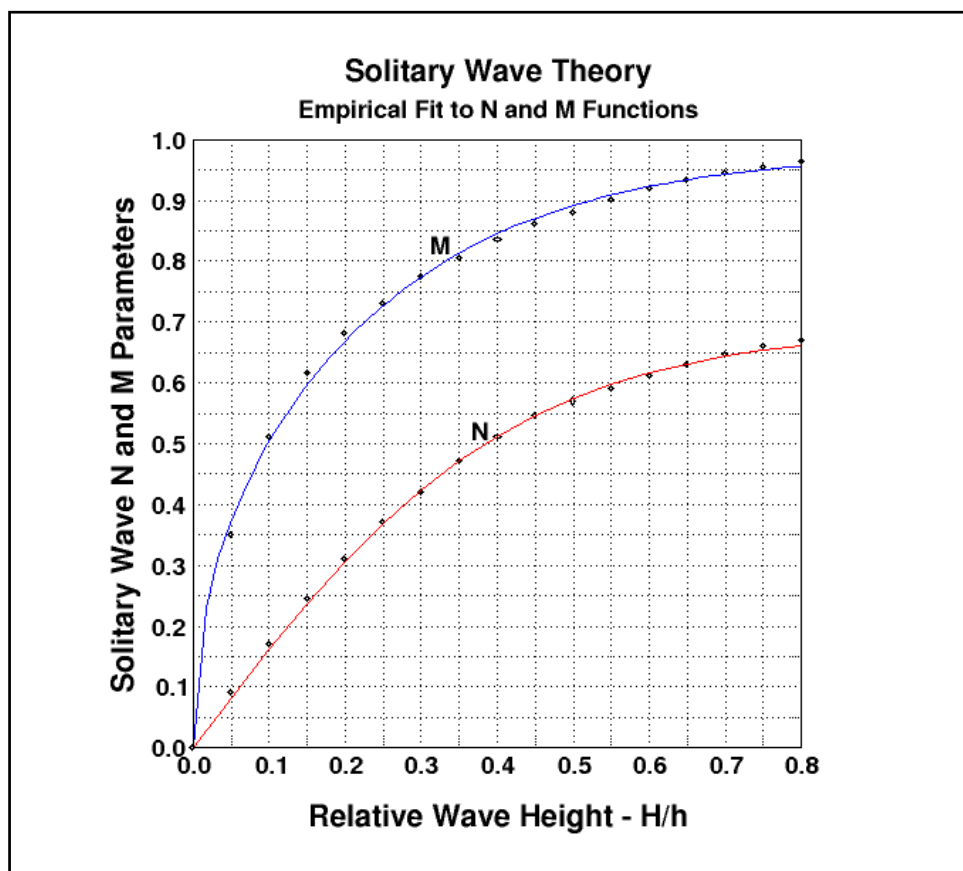


Figure 3. Nonlinear curve fit to solitary wave N and M coefficients

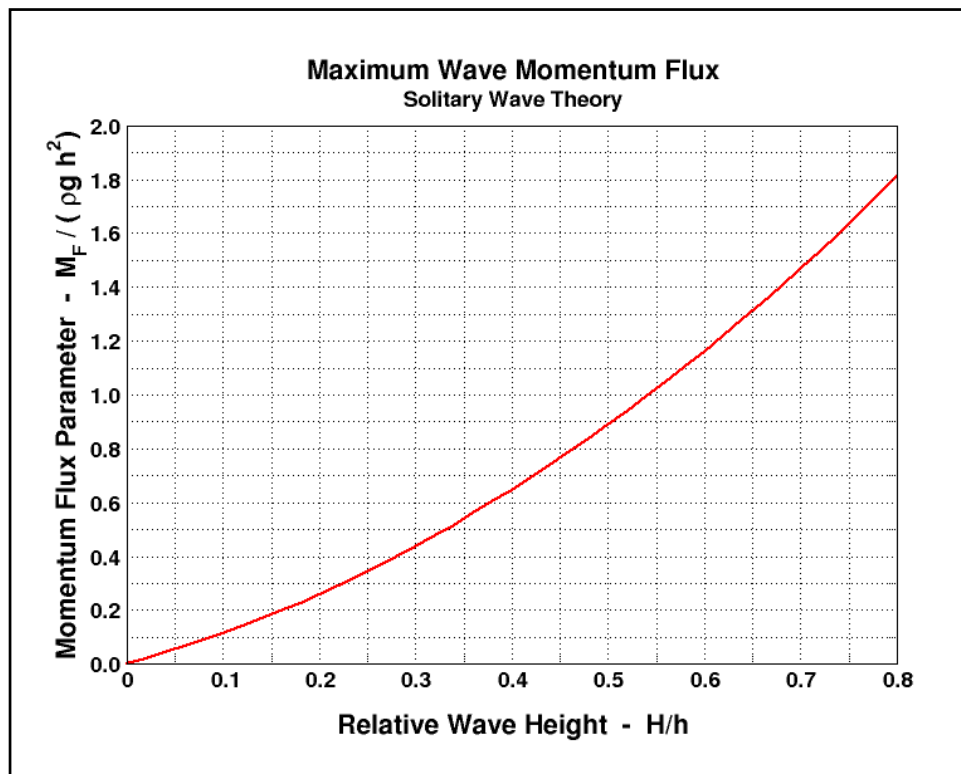


Figure 4. Solitary wave nondimensional wave momentum flux versus H/h

momentum flux parameter using existing runup for regular waves and solitary waves. CHETN-III-68 (Hughes 2003) extended this concept to irregular wave runup, providing improved estimation techniques for runup on smooth, impermeable slopes. In all runup cases, structure slope was found to have less influence than previously thought for the range of slopes in the data set.

EXAMPLE CALCULATIONS: The following examples illustrate calculation of the wave momentum flux parameter for regular and solitary waves.

Example: Regular Wave Momentum Flux Parameter

Find: The nondimensional wave momentum flux parameter and the corresponding maximum depth-integrated wave momentum flux per unit length of wave crest for the specified regular wave.

Given:

- | | |
|----------------------------------|--------------------------------|
| $h = 15$ ft | – Water depth |
| $T = 8$ s | – Regular wave period |
| $H = 9$ ft | – Regular wave height |
| $g = 32.2$ ft/s ² | – Gravitational acceleration |
| $\rho g = 64$ lb/ft ³ | – Specific weight of sea water |

Estimate Limit Wave Height: First check to be certain the specified wave height does not exceed the steepness limited wave height for the given wave period and water depth. Calculate the circular wave frequency and nondimensional depth as

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{(8 \text{ s})} = 0.7854 \text{ 1/s}$$

and

$$r = \frac{\omega^2 h}{g} = \frac{(0.7854 \text{ 1/s})^2 (15 \text{ ft})}{(32.2 \text{ ft/s}^2)} = 0.287$$

The limiting wave height is given by rearranging Equation 3 and replacing the constants with the given numerical values, i.e.,

$$H_{limit} = \left(\frac{g}{\omega^2} \right) (1.0575) \tanh \left(\frac{0.7879r + 2.0064r^2 - 0.0962r^3}{1 + 3.2924r - 0.2645r^2} \right)$$

Substituting for the variables ω , g , and r yields

$$H_{limit} = \left[\frac{32.2 \text{ ft/s}^2}{(0.7854 \text{ 1/s})^2} \right] (1.0575) \tanh \left[\frac{0.7879(0.287) + 2.0064(0.287)^2 - 0.0962(0.287)^3}{1 + 3.2924(0.287) - 0.2645(0.287)^2} \right]$$

or

$$H_{limit} = (55.2 \text{ ft}) \tanh (0.2023) = \underline{11.0 \text{ ft}}$$

Because $H < H_{limit}$, the specified wave condition is not steepness limited.

Calculate the Wave Momentum Flux Parameter: First calculate values of relative wave height and relative depth as

$$\frac{H}{h} = \frac{9 \text{ ft}}{15 \text{ ft}} = 0.6 \quad \text{and} \quad \frac{h}{gT^2} = \frac{15 \text{ ft}}{(32.2 \text{ ft/s}^2)(8 \text{ s})^2} = 0.0073$$

Next, find the values of the coefficient A_0 and A_1 from Equations 6 and 7, respectively as

$$A_0 = 0.639 \left(\frac{H}{h} \right)^{2.026} = 0.639 (0.6)^{2.026} = 0.2270$$

$$A_1 = 0.180 \left(\frac{H}{h} \right)^{-0.391} = 0.180 (0.6)^{-0.391} = 0.2198$$

Finally, the nondimensional wave momentum flux parameter is calculated from Equation 5 as

$$\left(\frac{M_F}{\rho g h^2} \right)_{\max} = A_0 \left(\frac{h}{g T^2} \right)^{-A_1} = 0.2270 (0.0073)^{-0.2198} = \underline{0.669}$$

Calculate the Maximum Depth-Integrated Wave Momentum Flux: The dimensional value of maximum depth-integrated wave momentum flux per unit length along the wave crest is obtained simply as

$$(M_F)_{\max} = (\rho g) (h^2) \left(\frac{M_F}{\rho g h^2} \right)_{\max} = (64 \text{ lb/ft}^3) (15 \text{ ft})^2 (0.669) = \underline{9,634 \text{ lb/ft}}$$

Example: Solitary Wave Momentum Flux Parameter

Find: The nondimensional wave momentum flux parameter and the corresponding maximum depth-integrated wave momentum flux per unit length of wave crest for the specified solitary wave.

Given:

- $h = 15 \text{ ft}$ – Water depth
- $H = 9 \text{ ft}$ – Solitary wave height
- $g = 32.2 \text{ ft/s}^2$ – Gravitational acceleration
- $\rho g = 64 \text{ lb/ft}^3$ – Specific weight of sea water

Calculate the Wave Momentum Flux Parameter: This solitary wave has the same relative wave height as the previous example, i.e.,

$$\frac{H}{h} = \frac{9 \text{ ft}}{15 \text{ ft}} = 0.6$$

Using this value of H/h , determine the coefficients M and N from Equations 9 and 10, respectively:

$$M = 0.98 \left\{ \tanh \left[2.24 \left(\frac{H}{h} \right) \right] \right\}^{0.44} = 0.98 \{ \tanh [2.24 (0.6)] \}^{0.44} = 0.923$$

and

$$N = 0.69 \tanh \left[2.38 \left(\frac{H}{h} \right) \right] = 0.69 \tanh [2.38 (0.6)] = 0.615$$

A quick check of the plots in Figure 3 with a value of $H/h = 0.6$ confirms the calculations.

Substituting values for H/h , M , and N into Equation 8 yields the nondimensional wave momentum flux parameter for the specified solitary wave, i.e.,

$$\begin{aligned} \left(\frac{M_F}{\rho g h^2} \right)_{\max} &= \frac{1}{2} \left[(0.6)^2 + 2(0.6) \right] \\ &\quad + \frac{(0.615)^2}{2(0.923)}(0.6+1) \left\{ \tan \left[\frac{(0.923)}{2}(0.6+1) \right] + \frac{1}{3} \tan^3 \left[\frac{(0.923)}{2}(0.6+1) \right] \right\} \\ \left(\frac{M_F}{\rho g h^2} \right)_{\max} &= 0.78 + 0.3278 \left[\tan(0.7384) + \frac{1}{3} \tan^3(0.7384) \right] = \underline{1.161} \end{aligned}$$

The plot shown in Figure 4 agrees with the previous calculation for a value of $H/h = 0.6$.

Caution: If you perform the previous calculation on a hand-held calculator, you may need to convert the arguments of the tangent function from radians to degrees before taking the tangent.

Calculate the Maximum Depth-Integrated Wave Momentum Flux: Just as for the regular wave example, the dimensional value of maximum depth-integrated wave momentum flux per unit length along the wave crest is calculated as

$$(M_F)_{\max} = (\rho g) (h^2) \left(\frac{M_F}{\rho g h^2} \right)_{\max} = (64 \text{ lb/ft}^3) (15 \text{ ft})^2 (1.161) = \underline{16,718 \text{ lb/ft}}$$

The value of $(M_F)_{\max}$ for the solitary wave is 1.74 times greater than the regular wave having the same value of H/h . In fact, the solitary wave is the limit of the regular wave as wave period becomes very long.

SUMMARY: This CHETN has described a new parameter representing the maximum depth-integrated wave momentum flux occurring in a wave. Because wave momentum flux has units of force per unit crest width, it is a physical descriptor of wave forces acting on coastal structures. For periodic waves an easily applied empirical expression was given to estimate nondimensional maximum depth-integrated wave momentum flux as a function of relative wave height and relative

water depth. A formula was also given for estimating the wave momentum flux parameter for solitary waves. Example calculations are provided for both wave types.

The wave momentum flux parameter is useful for developing improved semiempirical formulas to describe such wave/structure processes as runup, overtopping, reflection, transmission, and armor stability. The first application of the parameter has been revised design guidance for irregular wave runup on smooth, impermeable plane slopes as detailed in CHETN-III-68 (Hughes 2003).

ADDITIONAL INFORMATION: This CHETN is a product of the Scour at Inlet Structures Work Unit of the Coastal Inlets Research Program (CIRP) being conducted at the U.S. Army Engineer Research and Development Center, Coastal and Hydraulics Laboratory. Questions about this technical note can be addressed to Dr. Steven A. Hughes (Voice: 601-634-2026, Fax: 601-634-3433, email: Steven.A.Hughes@erdc.usace.army.mil). For information about the Coastal Inlets Research Program (CIRP), please contact the CIRP Technical Leader, Dr. Nicholas C. Kraus at Nicholas.C.Kraus@erdc.usace.army.mil. Beneficial reviews were provided by Mr. Dennis Markle and Dr. Jeff Melby, Coastal and Hydraulics Laboratory; and Mr. John Ahrens, retired Coastal and Hydraulics Laboratory and Sea Grant.

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